# Lightweight and Effective Sector Beam Pattern Synthesis with Uniform Linear Antenna Arrays

Joan Palacios, Danilo De Donno, and Joerg Widmer

***Abstract*—In this letter, we present a lightweight and effective method for the synthesis of sector beam patterns using uniform linear arrays. With the objective to approximate a desired array-factor response, we formulate an optimization problem which can be simplified and solved in closed form assuming real instead of complex array weights. As a solution to this problem, we derive a compact expression to compute the optimal array weights as a function only of the desired beamwidth and steering direction. Numerical experiments demonstrate that, compared to classical, state-of-the-art techniques, our solution can better approximate the target radiation mask, yet requires one order of magnitude lower computational complexity.**

***Index Terms*—Uniform linear arrays, antenna arrays, sector beam, beam pattern synthesis.**

## I. Introduction

The synthesis of sector beam patterns with antenna arrays is a widely investigated topic in the literature because of its myriad of applications, ranging from massive multiple-input multiple-output (MIMO) [1] to cell sectorization in cellular networks [2]. Very recently, the design of sector beam patterns has also received significant attention in the millimeter-wave (mmWave) context where the use of high-gain, adaptive antenna arrays with configurable beamwidth is essential to cope with the higher propagation loss and unfavorable atmospheric absorption at mmWave frequencies [3], [4].

Several approaches for sector beam synthesis have been proposed in the literature [5], [6], ranging from simple, classical formulations, e.g., Fourier transform, Woodward-Lawson frequency sampling, etc., derived as extensions of digital filter design techniques [7, Ch. 21] to more sophisticated methods such as genetic algorithms [8], particle swarm [9], and invasive weed [10]. One of the main problems of these techniques is their inherent complexity, which makes it more difficult for researchers and practitioners to implement and apply them. Secondly, since these techniques are quite computationally intensive, they may require many iterations until they convergence to the optimum solution. This last aspect is crucial, for example, in cognitive radios where the antenna configuration needs to be programmed dynamically based on the propagation environment and link performance to deliver the required quality of service [11]. In such situations, the availability of lightweight, effective, and fast algorithms for beam pattern synthesis is essential, especially in the case of resource-constrained, battery-powered wireless mobile nodes.

In this letter, we propose a new technique for the synthesis of sector beam patterns with uniform linear arrays (ULAs). The intuition behind the technique is that the problem of synthesizing an array factor with symmetrical, low-pass magnitude response can be significantly simplified by employing real instead of complex array weights. This simplification leads to a lightweight mathematical formulation whose solution yields a compact, closed-form expression to derive the antenna coefficients as a function only of the desired beamwidth and steering direction. The proposed strategy is more effective than classical approaches, not only in terms of ability to shape sector beam patterns which better comply with a desired mask, but also in terms of reduced computational complexity.

The letter is organized as follows. In Section II, the mathematical formulation for the proposed beam pattern synthesis technique is presented. Numerical assessment and comparisons with classical state-of-the-art synthesis techniques are provided in Section III. Finally, the main conclusions are drawn in Section IV.

## II. Formulation

We consider a wireless device equipped with a ULA of $M$ isotropic antenna elements, inter-element distance $d$, and operating wavelength $\lambda$. We focus on the digital wavenumber domain $\psi{=}(2\pi d/\lambda)\sin\phi$ referred to the azimuthal angle $\phi$ and consider the design of an ideal low-pass array factor centered at $\psi_0{=}0$ with beamwidth $\psi_\mathrm{b}$. As shown in Fig. 1, such an ideal response is defined over $-\pi \leq \psi \leq \pi$ by the following function:

$$\Pi(\psi) = \begin{cases} 1 & -\psi_\mathrm{b}/2 < \psi < \psi_\mathrm{b}/2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

In Fig. 1, we highlight also the three main radiation regions of a beam pattern, namely (i) the main-lobe region, which is defined by the mask beamwidth $\psi_\mathrm{b}$; (ii) the transition region, which is defined by the angular interval between the ideal mask boundary and the first pattern null; and (iii) the remaining side-lobe region.

We recall that the array factor for a ULA with steering direction $\psi_0{=}0$ and complex array weights $\mathbf{p}$ is given by:

$$AF(\psi) = \sum_{m=0}^{M-1} \mathbf{p}(m)\mathrm{e}^{-jm\psi} \quad (2)$$

The problem of designing a ULA with magnitude of the array factor $|AF(\psi)|$ as close as possible to the symmetrical

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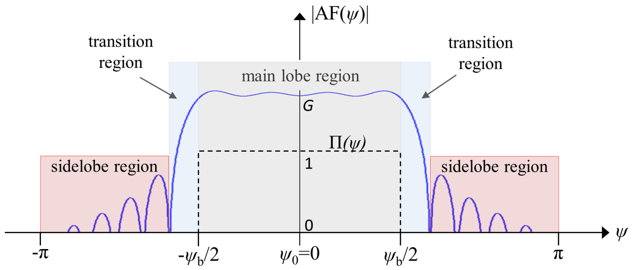



Fig. 1. Magnitude of the array factor: ideal objective mask and real sector beam pattern with the relevant regions of radiation highlighted.

ideal response $\Pi(\psi)$ can be significantly simplified by the following well-known lemma [12] which allows to consider real array weights $\mathbf{p} \in \mathbb{R}^M$ instead of complex array weights $\mathbf{p} \in \mathbb{C}^M$.

**Lemma 1.** *If it exists a set of complex array weights whose* $|AF(\psi)|$ *is symmetrical, then it exists a set of real array weights* $\mathbf{p}(m)$*, with* $m = 0, 1, ..., M-1$*, which gives rise to the same* $|AF(\psi)|$*.*

*Proof.* We start by taking the polynomial:

$$P(z) = \sum_{m=0}^{M-1} \mathbf{p}(m) z^m = a_0 \prod_{m=1}^{M} (z - z_m)$$

with roots $\{z_m\}_{m=1}^M$. Note that $AF(\psi) = P(\mathrm{e}^{-j\psi})$. Since $|AF(\psi)|$ is symmetrical, $|P(z)|^2 = |P(z^*)|^2$ in the complex circle $|z| = 1$. Moreover, since $|P(z)|^2$ and $|P(z^*)|^2$ are equal in a dense set, they are equal in the whole complex space, i.e. $|P(z)|^2 = |P(z^*)|^2, \forall z \in \mathbb{C}$. This means that if $z_0$ is a root of $P(z)$ with multiplicity $\alpha$, then it is a root of both $|P(z)|^2$ and $|P(z^*)|^2$ with multiplicity $2\alpha$ and, therefore, it is also a root of $P(z^*)$ with multiplicity $\alpha$. Finally, if $z_0$ is a root of $P(z)$ with multiplicity $\alpha$, then also $z_0^*$ is a root of $P(z)$ with multiplicity $\alpha$. This means that if we fix $a_0$ to be real, all the coefficients $\mathbf{p}(m)$, with $m = 0, 1, ..., M-1$, are real. □

Since the beam patterns to be synthesized are symmetrical, we can apply Lemma 1 and assume $\mathbf{p} \in \mathbb{R}^M$ in Eq. 2. The array factor expression can be therefore revised as:

$$AF(\psi) = \mathrm{e}^{j\beta\psi} \overline{AF}(\psi) \tag{3}$$

where $\beta = (M-1)/2$ and the term $\overline{AF}(\psi)$ is given by:

$$\overline{AF}(\psi) = \sum_{m=-\beta}^{\beta} \mathbf{p}(m+\beta) \mathrm{e}^{-jm\psi} \tag{4}$$

By Fourier analysis, $\overline{AF}(\psi)$ can be expressed as the sum of of sine and cosine functions with imaginary and real coefficients respectively. In order to further simplify the problem, we impose that all the sine terms in the Fourier expansion of $\overline{AF}(\psi)$ are zeroes. This leads to symmetrical antenna weights, i.e., $\mathbf{p}(i) = \mathbf{p}(j)$ for $i + j = 2\beta$ and $i, j \in \{0, 1, ..., M-1\}$. The reason behind this approximation relies on two main observations: (1) for the main steering direction, sine terms do not contribute to the radiated power because $\sin(\psi_0)$=0 for $\psi_0$=0; (2) neglecting the sine terms provides beam patterns with reduced sidelobe power level $\left|\overline{AF}(\psi)\right|^2_{\text{SL}}$ since:

$$\left|\overline{AF}(\psi)\right|^2_{\text{SL}} = |\mu_{\sin}(\psi)|^2 + |\nu_{\cos}(\psi)|^2 \geq |\nu_{\cos}(\psi)|^2$$

where $\mu_{\sin}(\psi)$ and $\nu_{\cos}(\psi)$ account for the sine and cosine terms in the Fourier expansion respectively. Finally, we can approximate $\overline{AF}(\psi)$ by the following sum of real terms for even $M$,

$$\overline{AF}(\psi) = 2 \sum_{m=\beta+\frac{1}{2}}^{M-1} \mathbf{p}(m) \cos[(m-\beta)\psi] \tag{5}$$

and for odd $M$,

$$\overline{AF}(\psi) = \mathbf{p}(\beta) + 2 \sum_{m=\beta+1}^{M-1} \mathbf{p}(m) \cos[(m-\beta)\psi] \tag{6}$$

Since $\overline{AF}(\psi)$ can be approximated by a real function built with the intent to not compromise the radiation in the main steering direction and, at the same time, minimize the sidelobe power level, we can assume $|AF(\psi)| \approx \overline{AF}(\psi)$. Therefore, the problem of designing a ULA with $|AF(\psi)|$ approximating the ideal response $\Pi(\psi)$ can be formulated as:

$$\begin{aligned} \mathbf{p} = & \arg\min_{\hat{\mathbf{p}}} \|\overline{AF}(\psi) - G\Pi(\psi)\| \\ & \text{s.t. } \overline{AF}(\psi) \text{ is defined by } \hat{\mathbf{p}}, \\ & 0 < G \in \mathbb{R}, \quad \|\hat{\mathbf{p}}\| = 1 \end{aligned} \tag{7}$$

or equivalently

$$\begin{aligned} \mathbf{p} = & \arg\min_{\hat{\mathbf{p}}} \|AF(\psi) - G\mathrm{e}^{j\beta\psi}\Pi(\psi)\| \\ & \text{s.t. } AF(\psi) \text{ is defined by } \hat{\mathbf{p}}, \\ & 0 < G \in \mathbb{R}, \quad \|\hat{\mathbf{p}}\| = 1 \end{aligned} \tag{8}$$

where $G$ is a positive variable accounting for the gain of the beam in the steering direction. The problem in Eq. 8 can be decomposed into two sub-problems, namely:

**Problem 1.** *Minimization over* $G$

$$\begin{aligned} \mathbf{p} = \mathbf{p}_G \text{ for } \quad G = & \arg\min_{G} \|AF_G(\psi) - G\mathrm{e}^{j\beta\psi}\Pi(\psi)\| \\ & \text{s.t. } AF_G(\psi) \text{ is defined by } \mathbf{p}_G, \\ & 0 < G \in \mathbb{R}, \quad \|\mathbf{p}_G\| = 1 \end{aligned} \tag{9}$$

**Problem 2.** *Minimization over* $\hat{\mathbf{p}}$ *for a given* $0 < G \in \mathbb{R}$

$$\begin{aligned} \mathbf{p}_G = & \arg\min_{\hat{\mathbf{p}}} \|AF(\psi) - G\mathrm{e}^{j\beta\psi}\Pi(\psi)\| \\ & \text{s.t. } AF(\psi) \text{ is defined by } \hat{\mathbf{p}}, \|\hat{\mathbf{p}}\| = 1 \end{aligned} \tag{10}$$

Considering $\mathbf{b}$ as the vector that contains the Fourier series coefficients of the objective function,

$$\mathbf{b}(m) = \mathcal{F}\left\{\mathrm{e}^{j\beta\psi}\Pi(\psi)\right\}(m) = \operatorname{sinc}\left[\frac{\psi_{\mathrm{b}}(m-\beta)}{2\pi}\right] \tag{11}$$

and applying Parseval's identity, Problem 2 can be reduced to the following problem:

$$\begin{aligned} \mathbf{p}_G = & \arg\min_{\hat{\mathbf{p}}} \|\hat{\mathbf{p}} - G\mathbf{b}\| + C_G \\ & \text{s.t. } AF(\psi) \text{ is defined by } \hat{\mathbf{p}}, \|\hat{\mathbf{p}}\| = 1 \end{aligned} \tag{12}$$

where $C_G$ can be removed from the optimization framework since it is a constant defined as:

$$C_G = \sum_{m<-\infty}^{-1} \left| \left[ \mathcal{F}\left( \mathrm{e}^{j\beta\psi}\Pi(\psi) \right) \right]_m \right|^2 + \sum_{m=M}^{+\infty} \left| G\left[ \mathcal{F}\left( \mathrm{e}^{j\beta\psi}\Pi(\psi) \right) \right]_m \right|^2$$

The problem in Eq. 12 represents the classical geometrical problem of finding the closest point to a multidimensional sphere with solution:

$$\mathbf{p}_G = \frac{G\mathbf{b}}{\|G\mathbf{b}\|} = \frac{\mathbf{b}}{\|\mathbf{b}\|} \tag{13}$$

Since $\mathbf{p}_G$ does not depend on $G$, the solution for Problem 1 is $\mathbf{p} = \frac{\mathbf{b}}{\|\mathbf{b}\|}$ as well. The array vector weights for any steering direction[1] $\psi_0 \neq 0$ can be easily obtained by multiplying each $\mathbf{b}(m)$ in Eq. 11 by $\mathrm{e}^{jm\psi_0}$, for $m = 0, 1, ..., M-1$.

## III. Results

In this section, we assess the performance of our formulation for the synthesis of sector beam patterns with ULAs and compare the results against those achieved with well-established state-of-the-art designs.

As a first numerical experiment, we synthesize a sector beam pattern defined by the purely rectangular mask $\Pi(\psi)$ in Fig. 1 with $\psi_\mathrm{b}$=0.9 rad (approximately 50°). Because of lack of space, we only consider the case of a ULA with $M$=32, $\lambda/2$-spaced isotropic antennas. In Fig. 2, we plot the beam patterns resulting from applying our formulation and three classical state-of-the-art designs for sector beam patterns, namely the Fourier Series Method with both rectangular (FSM-RW) and Kaiser (FSM-KW) windows, and the Woodward-Lawson frequency-sampling Method with Hamming window (WLM-HW) [7, Ch. 21]. Rectangular and Hamming windows can be configured in terms of desired beamwidth, but not in terms of stop-band attenuation (which is fixed, by definition, to 21 dB and 54 dB respectively). The Kaiser window, instead, is more customizable and allows the user to set, in addition to the beamwidth, the desired stop-band attenuation (set to 30 dB in this paper in order to provide a trade-off between rectangular and Hamming windows). As highlighted in Section II, the main strength of our design is its inherent simplicity. The main limitation of our solution is that it is not possible to set the required stop-band attenuation.

As shown in Fig. 2, compared with the other techniques, the proposed formulation provides comparable flatness in the main-lobe region and significantly reduced radiation in the transition region. In fact, in its transition region, the array factor of the proposed design decreases much more sharply than that of the other approaches. In addition to this, the proposed formulation has a smaller transition region compared with other techniques, with the first pattern nulls located at 0.56 rad, 0.65 rad, 0.78 rad, and 0.71 rad respectively for our design, FSM-RW, WLM-HW, and FSM-KW.

As a second numerical experiment, we analyze the percentage of power radiated by the beam patterns in the three different regions highlighted in Fig. 1. As evident from Fig. 3(a), independently from the beamwidth $\psi_\mathrm{b}$ to be synthesized, the proposed formulation significantly outperforms the other techniques in terms of ability to concentrate the radiated power in the main-lobe region. As a consequence of this, the power radiated by our beam patterns in the transition region (Fig. 3(b)) is drastically reduced in contrast with the other designs. The percentage of power radiated in the side-lobe region (Fig. 3(c)) is comparable for all the methods and, notably, it is always below 0.5% of the total radiated power for the proposed formulation. These aspects are paramount in directional wireless networks since excessive out-of-beam radiation is the primary source of interference to other devices.

[1]The interested reader can find the Matlab script of the proposed formulation at the following link: http://wireless.networks.imdea.org/software.

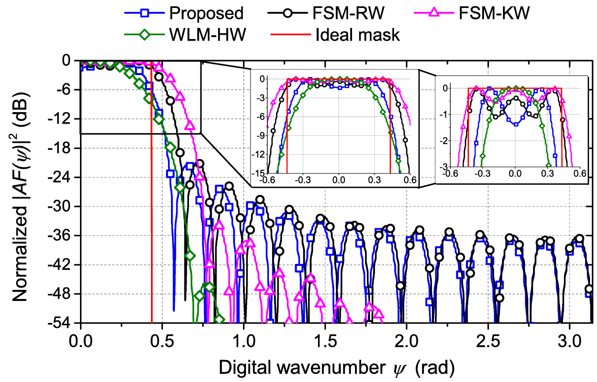


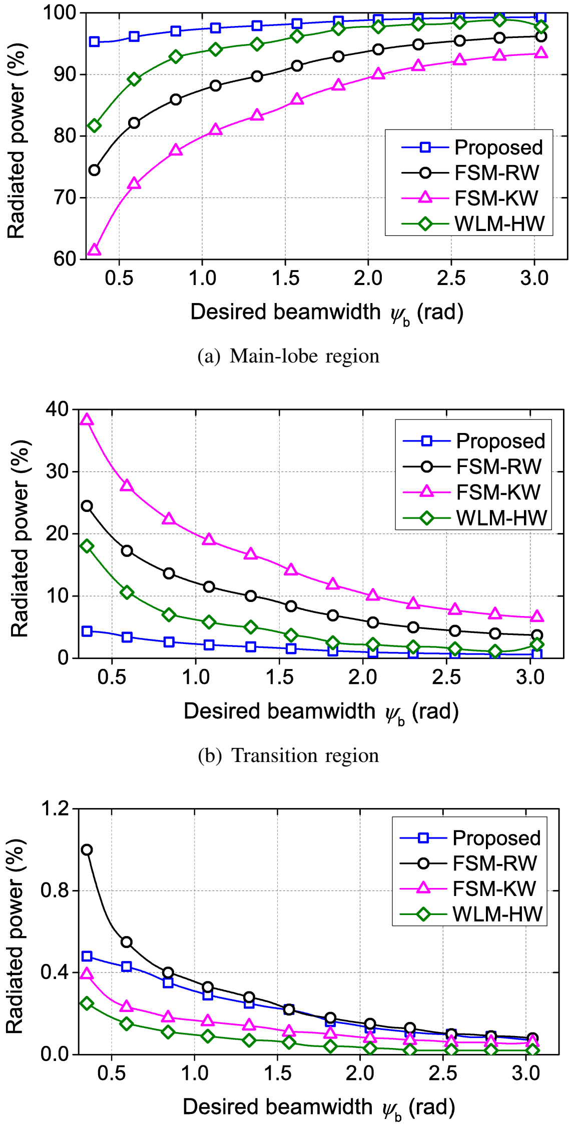

Fig. 2. Symmetric beam patterns synthesized with $M$=32 antenna elements for a desired beamwidth $\psi_\mathrm{b}$ = 0.9 rad (approximately 50°). Comparison between the proposed approach and state-of-the-art techniques.

In order to further assess the effectiveness of the proposed formulation to approximate the ideal mask, we calculate the half-power beamwidth (HPBW) error as $\Delta\psi_{3\mathrm{dB}} = |\psi_{3\mathrm{dB,synt}} - \psi_\mathrm{b}|$, i.e. as the absolute value of the difference between the HPBW of the synthesized beam pattern and the desired HPBW. The results plotted in Fig. 4 when varying the desired beamwidth $\psi_\mathrm{b}$ show that our beam patterns exhibit performance comparable with the FSM-RW. The $\Delta\psi_{3\mathrm{dB}}$ error obtained by our formulation is always below 0.1 rad, which is approximately 50% and 33% lower compared with that achieved by FSM-KW and WLM-HW respectively.

As a final numerical experiment, we analyze the run time required by the different design strategies to synthesize beam patterns with desired beamwidth $\psi_\mathrm{b}$=0.9 rad. In Fig. 5, we plot, as a function of the number of antenna elements, the computational times averaged over 500 Monte Carlo simulations on a PC with quad-core Intel Core i7 CPU. The results demonstrate the computational advantage of the proposed formulation which is approximately one order of magnitude faster than the other state-of-the-art techniques.

## IV. Conclusion

In this letter, we presented a new technique aimed at synthesizing sector beam patterns for ULAs. The proposed strategy is based on a very compact expression we derived for calculating the excitations of a ULA with radiation pattern conforming a target mask. Numerical experiments demonstrated that not only is our technique more effective than traditional methods




(a) Main-lobe region

(b) Transition region

(c) Side-lobe region

Fig. 3. Percentage of power radiated by the beam patterns synthesized with $M$=32 antennas in the different azimuthal regions defined in Fig. 1.

to shape sector beam patterns complying with the desired mask, but it also provides one order of magnitude lower computational complexity.

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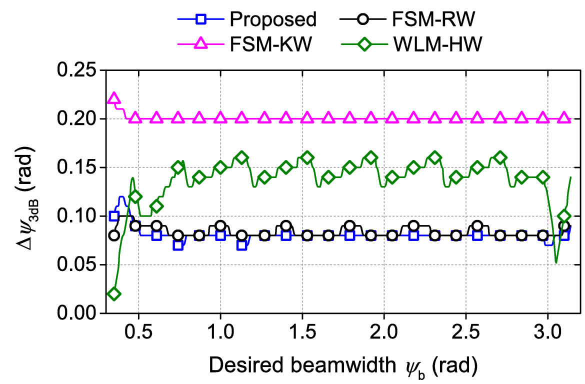


Fig. 4. HPBW error calculated as the absolute value of the difference between synthesized and desired HPBWs.

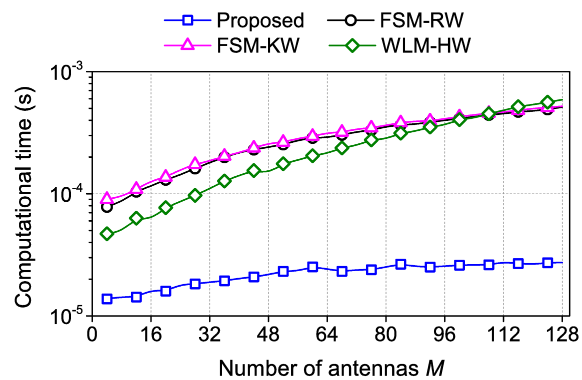


Fig. 5. Average computational time (on a semilogarithmic scale) required to synthesize beam patterns with beamwidth $\psi_b$=0.9 rad as a function of the number of antennas.

## References

[1] Y. Li, X. Ji, D. Liang, and Y. Li, "Dynamic beamforming for three-dimensional MIMO technique in LTE-Advanced networks," *International Journal of Antennas and Propagation*, vol. 2013, Jul. 2013.

[2] C. U. Saraydar and A. Yener, "Adaptive cell sectorization for CDMA systems," *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 6, pp. 1041–1051, Jun 2001.

[3] D. De Donno, J. Palacios, D. Giustiniano, and J. Widmer, "Hybrid analog-digital beam training for mmWave systems with low-resolution RF phase shifters," in *2016 IEEE ICC Workshop on 5G RAN Design*, May 2016.

[4] J. Palacios, D. De Donno, D. Giustiniano, and J. Widmer, "Speeding up mmWave beam training through low-complexity hybrid transceivers," in *27th Annual IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC '16)*, Sept. 2016.

[5] A. Massa, P. Rocca, and G. Oliveri, "Compressive sensing in electromagnetics - A review," *IEEE Antennas and Propagation Magazine*, vol. 57, no. 1, pp. 224–238, Feb 2015.

[6] P. Rocca, G. Oliveri, R. J. Mailloux, and A. Massa, "Unconventional phased array architectures and design methodologies - A review," *Proceedings of the IEEE*, vol. 104, no. 3, pp. 544–560, March, 2016.

[7] S. J. Orfinidas, "Electromagnetic waves and antennas," http://www.ece.rutgers.edu/~orfanidi/ewa/.

[8] F. J. Ares-Pena, J. A. Rodriguez-Gonzalez, E. Villanueva-Lopez, and S. R. Rengarajan, "Genetic algorithms in the design and optimization of antenna array patterns," *IEEE Transactions on Antennas and Propagation*, vol. 47, no. 3, pp. 506–510, Mar. 1999.

[9] M. M. Khodier and C. G. Christodoulou, "Linear array geometry synthesis with minimum sidelobe level and null control using particle swarm optimization," *IEEE Transactions on Antennas and Propagation*, vol. 53, no. 8, pp. 2674–2679, Aug. 2005.

[10] S. Karimkashi and A. A. Kishk, "Antenna array synthesis using invasive weed optimization: A new optimization technique in electromagnetics," in *2009 IEEE AP-S International Symposium*, Jun. 2009.

[11] M. H. Islam and Y. C. Liang, "Beam synthesis method for beamforming adaptation in cognitive radio based wireless communications systems," in *2007 IEEE Radio and Wireless Symposium*, Jan. 2007.

[12] S. Silver, *Microwave antenna theory and design*. McGraw-Hill, 1949.