Graphs of Edge-Intersecting Non-Splitting Paths in a Tree: Towards Hole Representations

(Extended Abstract)*

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Abstract. Given a tree and a set $\mathcal{P}$ of non-trivial simple paths on it, $\textsc{Vpt}(\mathcal{P})$ is the VPT graph (i.e. the vertex intersection graph) of $\mathcal{P}$, and $\textsc{Ept}(\mathcal{P})$ is the EPT graph (i.e. the edge intersection graph) of the paths $\mathcal{P}$ of the tree T. These graphs have been extensively studied in the literature. Given two (edge) intersecting paths in a graph, their *split vertices* is the set of vertices having degree at least 3 in their union. A pair of (edge) intersecting paths is termed *non-splitting* if they do not have split vertices (namely if their union is a path). In this work, we define the graph $\textsc{Enpt}(\mathcal{P})$ of edge intersecting non-splitting paths of a tree, termed the ENPT graph, as the (edge) graph having a vertex for each path in $\mathcal{P}$, and an edge between every pair of paths that are both edge-intersecting and non-splitting. A graph G is an ENPT graph if there is a tree T and a set of paths $\mathcal{P}$ of T such that $G = \textsc{Enpt}(\mathcal{P})$, and we say that $\langle T, \mathcal{P} \rangle$ is a *representation* of G. We show that trees, cycles and complete graphs are ENPT graphs. We characterize the representations of chordless ENPT cycles that satisfy a certain assumption. Unlike chordless EPT cycles which have a unique representation, these representations turn out to be multiple and have a more complex structure. Therefore, in order to give this characterization, we assume the EPT graph induced by the vertices of a chordless ENPT cycle is given, and we provide an algorithm that returns the unique representation of this EPT, ENPT pair of graphs. These representations turn out to have a more complex structure than chordless EPT cycles.

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1 Introduction

1.1 Background

Given a tree T and a set $\mathcal{P}$ of non-trivial simple paths in T, the Vertex Intersection Graph of Paths in a Tree (VPT) and the Edge Intersection Graph of Paths in a Tree (EPT) of $\mathcal{P}$ are denoted by $\text{VPT}(\mathcal{P})$ and $\text{EPT}(\mathcal{P})$, respectively. Both graphs have $\mathcal{P}$ as vertex set. $\text{VPT}(\mathcal{P})$ (resp. $\text{EPT}(\mathcal{P})$) contains an edge between two vertices if the corresponding two paths intersect in at least one vertex (resp. edge). A graph G is VPT (resp. EPT) if there exist a tree T and a set $\mathcal{P}$ of non-trivial simple paths in T such that G is isomorphic to $\text{VPT}(\mathcal{P})$ (resp. $\text{EPT}(\mathcal{P})$). In this case we say that $\langle T, \mathcal{P} \rangle$ is a VPT (resp. EPT) representation of G.

In this work we focus on edge intersections of paths, therefore whenever we are concerned with intersection of paths we omit the word "edge" and simply write that two paths *intersect*. We define a new class of graphs, called the graphs of edge intersecting and non-splitting paths of a tree (ENPT). Given a representation $\langle T, \mathcal{P} \rangle$ as described above, the related ENPT graph, denoted by $\text{ENPT}(\mathcal{P})$, has a vertex v for each path P_v of $\mathcal{P}$ and two vertices u, v of $\text{ENPT}(\mathcal{P})$ are adjacent if the paths P_u and P_v intersect and do not split (that is, their union is a path). A graph G is an ENPT graph if there is a tree T and a set of paths $\mathcal{P}$ of T such that G is isomorphic to $\text{ENPT}(\mathcal{P})$. We study the properties of this class ENPT.

We note that when T is a path $\text{EPT}(\mathcal{P}) = \text{ENPT}(\mathcal{P})$ is an Interval Graph. Therefore the class ENPT includes all Interval Graphs. Our aim is to study the structure of ENPT in order to classify them in the hierarchy of edge intersection graphs of paths in a tree.

EPT and VPT graphs have applications in communication networks. Assume that we model a communication network as a tree T and the message routes to be delivered in this communication network as paths on T. Two paths conflict if they both require to use the same link (vertex). This conflict model is equivalent to an EPT (a VPT) graph. Suppose we try to find a schedule for the messages such that no two messages sharing a link (vertex) are scheduled in the same time interval. Then a vertex coloring of the EPT (VPT) graph corresponds to a feasible schedule on this network. The motivation for the split condition for the paths can be summarized as follows: In optical networks, a router is an equipment responsible to route a message to a direction determined by its wavelength. So that two messages, corresponding to two splitting paths, can be correctly routed to different directions, they should be assigned two different wavelengths (see [REF] for more information).

EPT and VPT graphs have been extensively studied in the literature. Although VPT graphs can be characterized by a fixed number of forbidden subgraphs [13], it is shown that EPT graph recognition is NP-complete [11]. Edge intersection and vertex intersection give rise to identical graph classes in the case of paths in a line and in the case of subtrees of a tree. However, VPT graphs and EPT graphs are incomparable in general; neither VPT nor EPT contains the other. Main optimization and decision problems such as recognition [5], the maximum clique [6], the minimum vertex coloring [9] and the maximum stable set problems [14] are polynomial-time solvable in VPT whereas recognition and minimum vertex coloring problems remain NP-complete in EPT graphs [10]. In contrast, one can solve in polynomial time the maximum clique [11] and the maximum stable set [15] problems in EPT graphs.

After these works on EPT and VPT graphs in the early 80's, this topic did not get focus until very recently. Current research on intersection graphs is concentrated on the comparison of various intersection graphs of paths in a tree and their relation to chordal and weakly chordal graphs [7, 12]. Also, some tolerance model is studied via k-edge intersection graphs where two vertices are adjacent if their corresponding paths intersect on at least k edges [8]. Besides, several recent papers consider the edge intersection graphs of paths on a grid (e.g [1]).

1.2 Our Contribution

In this work we define the new family of ENPT graphs, and investigate its basic properties. We first study possible ENPT representations of some basic structures such as trees, cliques and holes. It should be noted that cliques play a crucial role in showing the NP-hardness of EPT graph recognition [10]. On the other hand, some forbidden subgraphs are determined in [11] using the fact that chordless cycles have a unique EPT representation, called a pie. It turns out that in ENPT graphs, representations of chordless cycles have a much more complicated structure, yielding several possible representations. In fact, given a chordless cycle C, several ENPT representations $\langle T, \mathcal{P} \rangle$ such that $\textsc{Enpt}(\mathcal{P})$ is isomorphic to C but $\textsc{Ept}(\mathcal{P})$ are non-isomorphic to each other are possible (see Figure 2).

Consider the pair (G, C) where G is a graph and C is a Hamiltonian cycle of G. We restrict our attention to the determination of a representation $\langle T, \mathcal{P} \rangle$ such that $\textsc{Ept}(\mathcal{P}) = G$ and $\textsc{Enpt}(\mathcal{P}) = C$. In this case we will say that $\langle T, \mathcal{P} \rangle$ is a representation of (G, C).

In Section 2 we give definitions, preliminaries and we provide ENPT representations of basic graphs such as trees, cliques and cycles. We also characterize all the ENPT representations of cliques. In Section 3 we obtain basic results regarding ENPT graphs, their relationship with EPT graphs and their representation. We then define the contraction operation, which is basically replacing two paths with their union provided that this union is a path. In Section 4 we introduce three assumptions and we characterize the representations of ENPT holes, i.e. representations $\langle T, \mathcal{P} \rangle$ for pairs (G, C), where C is a Hamiltonian cycle of G, such that $\textsc{Ept}(\mathcal{P}) = G$ and $\textsc{Enpt}(\mathcal{P}) = C$, satisfying these assumptions. In Section 5 we relax two out of these three assumptions, and extend the result of Section 4. Most of the proofs are either sketched or omitted in this Extended Abstract. The complete proofs appear in [2], except for Section 5 whose details can be found in [3].

2 Preliminaries and Basic Results

In this section we give definitions, present known related results, and develop basic results. The section is organized as follows: Section 2.1 is devoted to basic definitions, in Section 2.2 we present known results on EPT graphs and in Section 2.3 we present some basic ENPT graphs.

2.1 Definitions

General Notation: Given a graph G and a vertex v of G, we denote by $d_G(v)$ the degree of v in G. A vertex is called a *leaf* (resp. *intermediate vertex*, *junction*) if $d_G(v) = 1$ (resp. $= 2, \geq 3$). Whenever there is no ambiguity we omit the subscript G and write $d(v)$. Given a graph G, $\bar{V} \subseteq V(G)$ and $\bar{E} \subseteq E(G)$ we denote by $G[\bar{V}]$ and $G[\bar{E}]$ the subgraphs of G induced by $\bar{V}$ and by $\bar{E}$, respectively. The *union* of two graphs G, G' is the graph $G \cup G' \stackrel{def}{=} (V(G) \cup V(G'), E(G) \cup E(G'))$. The *join* $G + G'$ of two disjoint graphs G, G' is the graph $G \cup G'$ together with all the edges joining $V(G)$ and $V(G')$, i.e. $G + G' \stackrel{def}{=} (V(G) \cup V(G'), E(G) \cup E(G') \cup (V(G) \times V(G')))$. Given a (simple) graph G and $e \in E(G)$, we denote by $G_{/e}$ the (simple) graph obtained by contracting the edge $e = \{p, q\}$ of G, i.e. by coinciding the two endpoints of e to a single vertex $p.q$ and removing self loops and parallel edges. For any two vertices u, v of a tree T we denote by $p_T(u, v)$ the unique path between u and v in T. We denote the set of all positive integers at most k as $[k]$.

Intersections and union of paths: Given two paths P, P' in a graph, $P \parallel P'$ means that P and P' are non-intersecting, i.e. *edge-disjoint*. The *split vertices* of P and P' is the set of junctions in their union $P \cup P'$ and is denoted by $split(P, P')$. Whenever P and P' intersect and $split(P, P') = \emptyset$ we say that P and P' are *non-splitting* and denote this by $P \sim P'$. In this case $P \cup P'$ is a path or a cycle. When P and P' intersect and $split(P, P') \neq \emptyset$ we say that they are *splitting* and denote this by $P \nsim P'$. Clearly, for any two paths P and P' exactly one of the following holds: $P \parallel P'$, $P \sim P'$, $P \nsim P'$. When the graph G is a tree, the union $P \cup P'$ of two intersecting paths P, P' of G is a tree with at most two junctions, i.e. $|split(P, P')| \leq 2$ and $P \cup P'$ is a path whenever $P \sim P'$.

The VPT, EPT and ENPT graphs: Let $\mathcal{P}$ be a set of paths in a tree T. The graphs $\text{VPT}(\mathcal{P}), \text{EPT}(\mathcal{P})$ and $\text{ENPT}(\mathcal{P})$ are graphs such that $V(\text{ENPT}(\mathcal{P})) = V(\text{EPT}(\mathcal{P})) = V(\text{VPT}(\mathcal{P})) = \{p | P_p \in \mathcal{P}\}$. Given two distinct paths $P_p, P_q \in \mathcal{P}$, $\{p, q\}$ is an edge of $\text{ENPT}(\mathcal{P})$ if $P_p \sim P_q$, and $\{p, q\}$ is an edge of $\text{EPT}(\mathcal{P})$ (resp. $\text{VPT}(\mathcal{P})$) if P_p and P_q have a common edge (resp. vertex) in T. It follows that:

Remark 1. $E(\text{ENPT}(\mathcal{P})) \subseteq E(\text{EPT}(\mathcal{P})) \subseteq E(\text{VPT}(\mathcal{P}))$.

Two graphs G and G' such that $V(G) = V(G')$ and $E(G') \subseteq E(G)$ are termed a *pair (of graphs)* denoted as (G, G'). If $\text{EPT}(\mathcal{P}) = G$ (resp. $\text{ENPT}(\mathcal{P}) = G$) we say that $\langle T, \mathcal{P} \rangle$ is an EPT (resp. ENPT) representation for G. If $\text{EPT}(\mathcal{P}) = G$ and $\text{ENPT}(\mathcal{P}) = G'$ we say that $\langle T, \mathcal{P} \rangle$ is a representation for the pair (G, G'). Given a pair (G, G') the sub-pair induced by $\bar{V} \subseteq V(G)$ is the pair $(G[\bar{V}], G'[\bar{V}])$. Clearly, any representation of a pair induces representations for its induced sub-pairs.

Throughout this work, in all figures, the edges of the tree T of a representation $\langle T, \mathcal{P} \rangle$ are drawn as solid edges whereas the paths on the tree are shown by dashed edges. Similarly, edges of $\text{ENPT}(\mathcal{P})$ are drawn with solid or blue lines whereas edges in $E(\text{EPT}(\mathcal{P})) \setminus E(\text{ENPT}(\mathcal{P}))$ are dashed or red. We sometimes refer to them as blue and red edges, respectively. For an edge $e = \{p, q\}$ we use *split*(e) as a shorthand for *split*(P_p, P_q). We note that e is a red edge if and only if *split*$(e) \neq \emptyset$.

Cycles, Chords, Holes, Outerplanar graphs, Trees: Given a graph G and a cycle C of it, a *chord* of C in G is an edge of $E(G) \setminus E(C)$ connecting two vertices of $V(C)$. The *length* of a chord connecting the vertices i,j is the length of a shortest path between i and j on C. C is a *hole* (chordless cycle) of G if G does not contain any chord of C. This is equivalent to saying that the subgraph $G[V(C)]$ of G induced by the vertices of C is a cycle. For this reason a chordless cycle is also called an *induced* cycle.

An *outerplanar* graph is a planar graph that can be embedded in the plane such that all its vertices are on the unbounded face of the embedding. An outerplanar graph is Hamiltonian if and only if it is biconnected; in this case the unbounded face forms the unique Hamiltonian cycle. The *weak dual* graph of a planar graph G is the graph obtained from its dual graph by removing the vertex corresponding to the unbounded face of G. The weak dual graph of an outerplanar graph is a forest, and in particular the weak dual graph of a Hamiltonian outerplanar graph is a tree [4].

2.2 Preliminaries on EPT graphs

We now present definitions and results from [11]. A *pie* of a representation $\langle T, \mathcal{P} \rangle$ of an EPT graph is an induced star $K_{1,k}$ of T with k leaves $v_0, v_1, \ldots, v_{k-1} \in V(T)$, and k paths $P_0, P_1, \ldots P_{k-1} \in \mathcal{P}$, such that for every $0 \leq i \leq k-1$ both v_i and $v_{(i+1) \textbf{ mod } k}$ are vertices of P_i. We term the central vertex of the star as the *center* of the pie. It is easy to see that the EPT graph of a pie with k leaves is the hole C_k on k vertices. Moreover, this is the only possible EPT representation of C_k when $k \geq 4$:

Theorem 1. *[11] If an* EPT *graph contains a hole with $k \geq 4$ vertices, then every representation of it contains a pie with k paths.*

Let $\mathcal{P}_e \stackrel{def}{=} \{p \in \mathcal{P} | \ e \in p\}$ be the set of paths in $\mathcal{P}$ containing the edge e. A star $K_{1,3}$ is termed a *claw*. For a claw K of a tree T, $\mathcal{P}[K] \stackrel{def}{=} \{p \in \mathcal{P} | \ p \text{ uses two edges of } K\}$. It is easy to see that both $\text{EPT}(\mathcal{P}_e)$ and $\text{EPT}(\mathcal{P}[K])$ are cliques. These cliques are termed *edge clique* and *claw clique*, respectively. Moreover, these are the only possible representations of cliques.

Theorem 2. *[11] Any maximal clique of an* EPT *graph with representation $\langle T, \mathcal{P} \rangle$ corresponds to a subcollection $\mathcal{P}_e$ of paths for some edge e of T, or to a subcollection $\mathcal{P}[K]$ of paths for some claw K of T.*

2.3 Some ENPT graphs

In this section we show that trees, cycles and cliques are ENPT graphs, and give a complete characterization of the ENPT representations of cliques:

Lemma 1. *Every clique K of* $\textsc{Enpt}(\mathcal{P})$ *corresponds to an edge clique of* $\textsc{Ept}(\mathcal{P})$*, such that the union of the paths representing K is a path.*

A direct consequence of Lemma 1 is that the maximum clique problem in ENPT graphs can be solved in polynomial time. As there are at most $O(V(T)^3)$ maximal cliques in G, a maximum clique can be found using a clique enumeration algorithm, e.g. [16].

Lemma 2. *Every tree is an* ENPT *graph.*

Let T be a tree with k leaves and $\pi = (\pi_0, \ldots, \pi_{k-1})$ a cyclic permutation of the leaves. The *tour* (T, π) is the following set of $2k$ paths: (T, π) contains k *long* paths, each of which connecting two consecutive leaves $\pi_i, \pi_{i+1 \textbf{ mod } k}$. (T, π) contains k *short* paths, each of which connecting a leaf π_i and its unique neighbor in T (see Figure 1-c). Note that $\text{ENPT}((T, \pi))$ is a cycle.

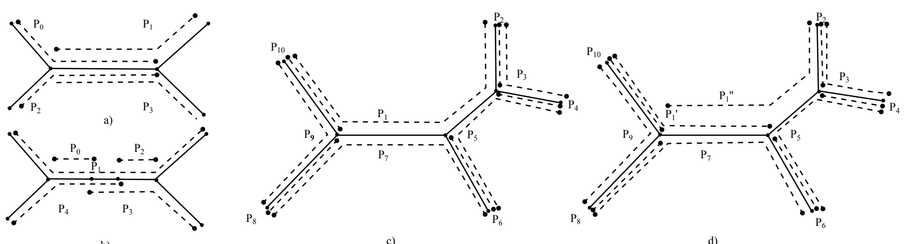

Fig. 1. a) A minimal representation of C_4 b) A minimal representation of C_5 c) A tour representation of the even hole C_{10}, d) A representation of the odd hole C_{11}

A *planar embedding* of a tour is a planar embedding of the underlying tree such that any two paths of the tour do not cross each other. A tour is *planar* if there exists a planar embedding of it. The tour in Figure 1-c is a planar embedding of a tour. Note that a tour (T, π) is planar if and only if π corresponds to the order in which the leaves are encountered by some DFS traversal of T.

Lemma 3. *Every cycle C_k is an* ENPT *graph.*

Proof. $C_3 = K_3$ is an ENPT graph by Lemma 1. As for C_4 and C_5, possible ENPT representations are shown in Figure 1-(a,b), respectively. Any even hole C_{2k}, $(k \geq 3)$ is an ENPT graph. Indeed, for any tree T with k leaves, and a cyclic permutation π of its leaves, the tour (T, π) constitutes an ENPT representation of C_{2k}. Any odd hole C_{2k+1}, $(k \geq 3)$ is an ENPT graph. Let T be a tree with k leaves. Split any long path of some tour (T, π) into two intersecting sub-paths such that no chord is created (if necessary subdivide an edge of the tree into two edges) (see Figure 1-d). The set of $2k+1$ paths obtained in this way constitutes an ENPT representation for C_{2k+1}. $\square$

3 Representations of EPT, ENPT Pairs: Basic Properties

In this section we develop the basic tools towards our goal of characterizing representations of $\mathrm{EPT}, \mathrm{ENPT}$ pairs. We define an equivalence relation and a partial order on representations. In this work, we focus on finding representations that are minimal with respect to this partial order. We define the contraction operation on pairs, and the union operation on representations. The contraction operation is a restricted variant of graph contraction operation that operates on both graphs of a pair. The union operation is the operation of replacing two paths by their union whenever possible.

Equivalent and minimal representations: We say that the representations $\langle T_1, \mathcal{P}_1 \rangle$ and $\langle T_2, \mathcal{P}_2 \rangle$ are *equivalent*, and denote it by $\langle T_1, \mathcal{P}_1 \rangle \approxeq \langle T_2, \mathcal{P}_2 \rangle$, if their corresponding EPT and ENPT graphs are isomorphic under the same isomorphism (in other words, if they constitute representations of the same pair of graphs (G, G')).

We write $\langle T_2, \mathcal{P}_2 \rangle \lesssim \langle T_2, \mathcal{P}_2 \rangle$ if $\langle T_2, \mathcal{P}_2 \rangle$ can be obtained from $\langle T_1, \mathcal{P}_1 \rangle$ by successive application of (one of) the following *minifying* operations: a) Contraction of an edge e of T_1 (and of all the paths in $\mathcal{P}_1$ using e), b) Removal of an initial edge (*tail*) of a path in $\mathcal{P}_1$. $\langle T, \mathcal{P} \rangle$ is a *minimal* representation if it is minimal in the partial order $\lesssim$ restricted to the representations representing the same pair. Throughout the work we aim at characterizing minimal representations.

EPT Holes: The ENPT graph of a pie is an independent set. Therefore

Remark 2. A hole of size at least 4 of an EPT graph does not contain blue (i.e. ENPT) edges.

Combining with Theorem 1, we obtain the following characterization of pairs (C_k, G'):

- $k > 3$. In this case C_k is represented by a pie. Therefore G' is an independent set. In other words, C_k consists of red edges. We term such a hole a red hole.
- $k = 3$ and C_k consists of red edges (G' is an independent set). We term such a hole a red triangle.
- $k = 3$ and C_k contains exactly one ENPT (blue) edge ($G' = P_1 \cup P_2$). We term such a hole a BRR triangle, and its representation is an edge clique.
- $k = 3$ and C_k contains two ENPT (blue) edges ($G' = P_3$). We term such a hole a BBR triangle, and its representation is an edge clique.
- $k = 3$ and C_k consists of blue edges ($G' = C_3$). We term such a hole a blue triangle.

EPT contraction: Let $\langle T, \mathcal{P} \rangle$ be a representation and $P_p, P_q \in \mathcal{P}$ such that $P_p \sim P_q$. We denote by $\langle T, \mathcal{P} \rangle_{/P_p, P_q}$ the representation that is obtained from $\langle T, \mathcal{P} \rangle$ by replacing the two paths P_p, P_q by the path $P_p \cup P_q$, i.e. $\langle T, \mathcal{P} \rangle_{/P_p, P_q} \stackrel{def}{=} \langle T, \mathcal{P} \setminus \{P_p, P_q\} \cup \{P_p \cup P_q\} \rangle$. We term this operation a *union*, and note the following important property of split vertices with respect to the union operation, and the following Lemma that it implies.

Remark 3. For every $P_p, P_q, P_r \in \mathcal{P}$ such that $P_p \sim P_q$, $split(P_p \cup P_q, P_r) = split(P_p, P_r) \cup split(P_q, P_r)$.

Lemma 4. *Let $\langle T, \mathcal{P} \rangle$ be a representation for the pair (G, G'), and let $e = \{p, q\} \in E(G')$. Then $G_{/e}$ is an* EPT *graph. Moreover $G_{/e} = \text{EPT}(\langle T, \mathcal{P} \rangle_{/P_p, P_q})$.*

We now extend the definition of the contraction operation to pairs. Based on Observation 3, the contraction of an ENPT edge does not preserve ENPT edges. More concretely, let P_p,P_q and $P_{q'}$ such that $P_p \sim P_q$, $P_p \sim P_{q'}$ and $P_q \nsim P_{q'}$. Then $G'_{/p,q}$ is not necessarily isomorphic to $\text{ENPT}(\langle T, \mathcal{P} \rangle_{/P_p, P_q})$ as $\{q', p.q\} \notin E(\text{ENPT}(\langle T, \mathcal{P} \rangle_{/P_p, P_q}))$. Let (G, G') be a pair and $e \in E(G')$. If for every edge $e' \in E(G')$ incident to e, the edge $e'' = e \triangle e'$ (forming a triangle together with e and e') is not an edge of G then $(G, G')_{/e} \stackrel{def}{=} (G_{/e}, G'_{/e})$, otherwise $(G, G')_{/e}$ is undefined. Whenever $(G, G')_{/e}$ is defined we say that (G, G') is *contractible* on e, or that e is *contractible*. A pair (G, G') is *contractible* if it contains at least one contractible edge. Clearly, (G, G') is non-contractible if and only if every edge of G' is contained in at least one BBR triangle.

4 Representation of ENPT Holes

The ENPT representations of C_3 is characterized by Lemma 1. Therefore we assume $n > 3$, which implies that (G, C_n) does not contain blue triangles. Moreover, in this section we confine ourselves to pairs (G, C_n) and representations $\langle T, \mathcal{P} \rangle$ satisfying the following three assumptions:

$(P1)$: (G, C_n) is not contractible.
$(P2)$: (G, C_n) is (K_4, P_4)-free, i.e., it does not contain an induced sub-pair isomorphic to a (K_4, P_4).
$(P3)$: Every red triangle of (G, C_n) is a claw clique, i.e. corresponds to a pie of $\langle T, \mathcal{P} \rangle$.

Assumptions $(P1), (P2)$ are relaxed in Section 5. Note that $(P1)$ and $(P2)$ are assumptions about the pair (G, C) and $(P3)$ is an assumption about the representation $\langle T, \mathcal{P} \rangle$. We say that $(P3)$ holds for a pair (G, C) if it has a representation $\langle T, \mathcal{P} \rangle$ satisfying $(P3)$.

W.l.o.g. let $V(G) = V(C_n) = \{0, 1, \ldots, n-1\}$ where the numbering of the vertices follows their order in C. Arithmetic operations on vertex numbers are modulo n. The corresponding set of paths is $\mathcal{P} = \{P_0, \ldots, P_{n-1}\}$.

C_4 is exceptional because all its representations satisfy assumptions $(P1-3)$, but some of our results fail to hold. The following Lemma 5 characterizes the representations of (G, C_4).

Lemma 5. *(i) All the representations of (G, C_4) satisfy assumptions $(P1 - 3)$, (ii) G is one of the two graphs in Figure 2, and (iii) each one of these two graphs has a unique minimal representation (also depicted in Figure 2).*

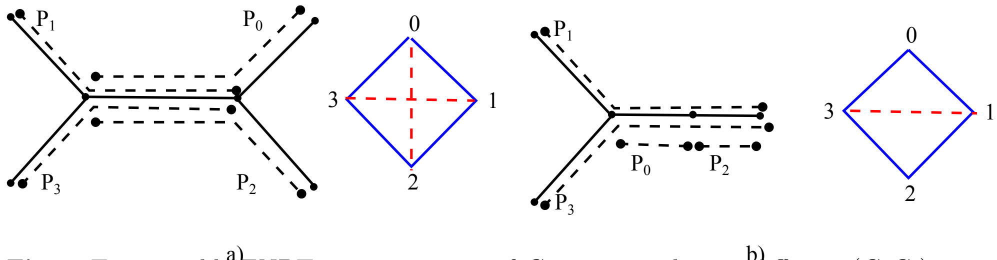

Fig. 2. Two possible ENPT representations of C_4 corresponding to different (G, C_4) pairs.

Weak Dual Trees:

We extend the definition of the weak dual tree of Hamiltonian outerplanar graphs to any Hamiltonian graph as follows. Given a pair (G, C) where C is a Hamiltonian cycle of G, a weak dual tree of (G, C) is the weak dual tree $\mathcal{W}(G, C)$ of an arbitrary Hamiltonian maximal outerplanar subgraph $\mathcal{O}(G, C)$ of G. $\mathcal{O}(G, C)$ can be built by starting from C and adding to it arbitrarily chosen chords from G as long as such chords exists and the resulting graph is planar.

Vertices of $\mathcal{W}(G, C)$ correspond to faces of $\mathcal{O}(G, C)$, and the faces of $\mathcal{O}(G, C)$ correspond to holes of G. The degree of a vertex of $\mathcal{W}(G, C)$ is the number of red edges in the corresponding face of $\mathcal{O}(G, C)$. To emphasize the difference, for an outerplanar graph G we will refer to *the* weak dual tree of G, whereas for a (not necessarily outerplanar) graph G we will refer to *a* weak dual tree of G.

Edges of $\mathcal{W}(G, C)$ correspond to red edges of $\mathcal{O}(G, C)$. The degree of a vertex of $\mathcal{W}(G, C)$ is the number of red edges in the corresponding face of $\mathcal{O}(G, C)$. Therefore leaves (resp. intermediate vertices, junctions) of $\mathcal{W}(G, C)$ correspond to BBR triangles (resp. BRR triangles, red holes) of (G, C). $|V(G)| = |V(C)| = |E(C)| = 2\ell + i$ where ℓ is the number of leaves of $\mathcal{W}(G, C)$ and i is the number of its intermediate vertices.

Lemma 6. *Let $n > 4$ and (G, C_n) be a pair satisfying $(P1-3)$. Then every edge of C_n is in exactly one BBR triangle.*

Lemma 7. *Let (G, C) be a pair satisfying $(P2), (P3)$ and let $\mathcal{W}(G, C)$ be a weak dual tree of (G, C). (i) There is a bijection between the contractible edges of (G, C) and the intermediate vertices of $\mathcal{W}(G, C)$. (ii) The tree obtained from $\mathcal{W}(G, C)$ by smoothing out the intermediate vertex corresponding to a contractible edge e is a weak dual tree of $(G, C)_{/e}$.*

We note that Lemma 6 does not hold for $n = 4$. However the following corollary of lemmata 6 and 7 holds for every n.

Corollary 1. *If (G, C) is a pair satisfying $(P1-3)$ with C isomorphic to C_n, then: (i) $\mathcal{W}(G, C)$ does not have intermediate vertices, (ii) n is even and $\mathcal{W}(G, C)$ has $n/2$ leaves. (iii) $\mathcal{W}(G, C)$ is a path if and only if $n = 4$.*

The Minimal Representation: Algorithm 1 gets a pair (G, C) satisfying assumptions $(P1), (P2)$ where C is a (Hamiltonian) cycle of G, and returns a planar tour that is the unique minimal representation of (G, C) satisfying $(P3)$.

The algorithm finds a planar tour of a weak dual tree $\mathcal{W}(G, C)$, and verifies that the solution found is valid before returning it, otherwise it returns that no solution exists. The representation $\langle \bar{T}, \bar{\mathcal{P}} \rangle$ calculated by the algorithm is a planar tour, that clearly satisfies $(P3)$. If (G, C) has no representation satisfying $(P3)$, then the algorithm detects this at line 10 and returns correctly that there is no solution. Therefore, we assume that (G, C) has at least one representation satisfying $(P3)$. The correctness is implied by the following lemma.

Algorithm 1 BUILDPLANARTOUR(G, C)

Require: $|V(G)| \geq 5$, (G, C) satisfies $(P1), (P2)$
1: $\bar{T} \leftarrow \mathcal{W}(G, C)$. ▷ Corresponding to $\mathcal{O}(G, C)$
Build the planar tour:
2: Let $\{v_0, v_1, \ldots, v_{k-1}\}$ be the leaves of $\bar{T}$ ordered by the DFS traversal of $\bar{T}$
3: corresponding to the planar embedding suggested by $\mathcal{O}(G, C)$.
4: Let $L_i = p_{\bar{T}}(v_i, v_{i+1 \textbf{ mod } k})$
5: Let S_i be the path of length 1 starting at v_i.
6: $\bar{\mathcal{P}}_L \leftarrow \{L_i | \ 0 \leq i \leq n-1\}$, $\bar{\mathcal{P}}_S \leftarrow \{S_i | \ 0 \leq i \leq n-1\}$.
7: Let $\bar{P}_i = \begin{cases} L_{i/2} & \text{if } i \text{ is even} \\ S_{\lfloor i/2 \rfloor} & \text{otherwise} \end{cases}$
8: $\bar{\mathcal{P}} \leftarrow \left\{ \bar{P}_i | \ 0 \leq i \leq 2n-1 \right\}$ ▷ $= \bar{\mathcal{P}}_L \cup \bar{\mathcal{P}}_S$
9:
10: **if** EPT$(\bar{\mathcal{P}}) = G$ **then return** $\langle \bar{T}, \bar{\mathcal{P}} \rangle$
11: **else return** "NO SOLUTION"
12: **end if**

Lemma 8. *Let (G, C) be a pair satisfying $(P1-3)$, $\langle T, \mathcal{P} \rangle$ a representation of (G, C) satisfying $(P3)$ and $\langle \bar{T}, \bar{\mathcal{P}} \rangle$ the representation returned by the algorithm. Then $\langle \bar{T}, \bar{\mathcal{P}} \rangle \cong \langle T, \mathcal{P} \rangle$ and $\langle \bar{T}, \bar{\mathcal{P}} \rangle \lesssim \langle T, \mathcal{P} \rangle$.*

Sketch of proof: For a representation $\langle T, \mathcal{P} \rangle$ of (G, C) that satisfies $(P3)$ we define a mapping $f : V(\bar{T}) \mapsto V(T)$ that maps junctions to junctions. The basic property of this mapping is that for a given vertex u of $\mathcal{W}(G, C)$, and every vertex i on the corresponding face of $\mathcal{O}(G, C)$, the vertex $f(u)$ is on the path P_i.

A junction u of $\bar{T}(= \mathcal{W}(G, C))$ corresponds to a face of $\mathcal{O}(G, C)$ which in turn corresponds to a hole of G corresponding to a pie of $\langle T, \mathcal{P} \rangle$. $f(u)$ is the center of this pie. A leaf v of $\bar{T}$ is adjacent to a junction u. v corresponds to a BBR triangle $\{i-1, i, i+1\}$ of $\mathcal{O}(G, C)$. Then $\{i-1, i+1\}$ is a red edge of G belonging to the face in $\mathcal{O}(G, C)$ corresponding to a pie centered at $f(u)$. Therefore P_{i-1} and P_{i+1} are two consecutive paths of this pie, i.e. $f(u) \in split(P_{i-1}, P_{i+1})$ and the paths P_{i-1}, P_{i+1} intersect on some path P of T starting at $f(u)$. The path P_i satisfies $P_i \sim P_{i-1}$ and $P_i \sim P_{i+1}$, therefore it intersects the path P. $f(v)$ is the most distant vertex from $f(u)$ on this intersection.

We prove that f preserves the topology of the tree. We then define a set of paths $\mathcal{P}^*$ of the minimum subtree T^* of T containing all the vertices $\left\{ f(u) | u \in \bar{T} \right\}$ such that $\langle T^*, \mathcal{P}^* \rangle$ is equivalent to $\langle \bar{T}, \bar{\mathcal{P}} \rangle$ and $\langle \bar{T}, \bar{\mathcal{P}} \rangle \lesssim \langle T^*, \mathcal{P}^* \rangle \lesssim \langle T, \mathcal{P} \rangle$. □

We are now ready to prove our main result

Theorem 3. *If $n > 4$ the following statements are equivalent:*
(i) (G, C_n) satisfies assumptions $(P1-3)$.
(ii) (G, C_n) has a unique minimal representation satisfying $(P3)$ which is a planar tour of a weak dual tree of G.
(iii) G is Hamiltonian outerplanar and every face adjacent to the unbounded face F is a triangle having two edges in common with F, (i.e. a BBR triangle).

Proof. (i) $\Rightarrow$ (ii) Implied by Lemma 8.
(ii) $\Rightarrow$ (iii) Consider a planar tour representation $\langle T, \mathcal{P} \rangle$. We show that $\mathrm{EPT}(\mathcal{P})$ is a Hamiltonian outerplanar graph. As $\mathcal{P}$ is a tour, $\mathrm{ENPT}(\mathcal{P})$ is a ring, therefore $\mathrm{EPT}(\mathcal{P})$ is Hamiltonian. It is not hard to prove that no chords of this cycle are crossing.

(iii) $\Rightarrow$ (i) Assume that G is outerplanar with faces adjacent to the unbounded face being BBR triangles. Consequently G is K_4-free, thus satisfies $(P2)$. Moreover every edge of C is in (exactly) one BBR triangle, therefore $(P1)$ holds. The planar tour of the weak dual tree of G is a representation of (G, C). This representation satisfies $(P3)$ because every edge clique of size 3 contains one short path whose incident edges are blue. □

5 Extensions

The details of the results presented in this section are given in [3]. When we relax assumption $(P1)$ then the unique minimal representation can be obtained by slightly modifying the planar tour as follows. Let us call *breaking apart* the inverse of a sequence of union operations that create one path. A *broken tour* is a representation obtained by breaking apart long paths of a tour.

Theorem 4. *[3] Let (G, C) be a pair satisfying $(P2), (P3)$. The unique minimal representation $\langle T', \mathcal{P}' \rangle$ of (G, C) satisfying $(P3)$ is a broken planar tour. Moreover $\langle T', \mathcal{P}' \rangle$ can be calculated in polynomial-time.*

We further relax assumption $(P2)$ and we replace all sub-pairs (K_4, P_4) by BBR triangles. The unique minimal representation of the modified pair is a broken planar tour by Theorem 4. We replace the short paths (corresponding to the inserted BBR triangles) by two paths in an appropriate way. We call such a representation a *broken planar tour with cherries*.

Theorem 5. *[3] The minimum representation $\langle T, \mathcal{P} \rangle$ satisfying $(P3)$ of a pair (G, C) is an broken planar tour with cherries. Moreover $\langle T', \mathcal{P}' \rangle$ can be calculated in polynomial-time.*

Generalization of the results to representations that do not satisfy assumption $(P3)$ is work in progress. Note that if we allow red edge cliques in the representation $\langle T, \mathcal{P} \rangle$ then $\langle T, \mathcal{P} \rangle$ is not necessarily a planar tour, as any tour is a representation of a hole.

Another direction of research would be to investigate the relation of the class of ENPT graphs with other graph classes, in particular with EPT.

ENPT\EPT $\neq \emptyset$ because the wheel graph $W_{5,1} = C_5 + K_1$ is in ENPT\EPT. In [11] graphs in VPT$\cap$EPT are characterized. The characterization of the graphs in EPT $\cap$ ENPT is an interesting research topic. Lastly, decision/optimization problems restricted to ENPT graphs, such as minimum vertex coloring, maximum stable set, and hardness of recognition of ENPT graphs seem to be major problems to investigate.

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